Meeting 7 — Math 22

Midterm Exam:

- Released at 3:30 pm today via canvas.
- Available until 11:59pm on 7/10.
- Timed 135 minutes
  1. 90 minutes to take exam
  2. 45 minutes to scan/submit your work
  3. Start by 9:44 pm on 7/10. The quiz closes and automatically submits at 11:59 pm on 7/10.
- Submit via canvas.
- You can only submit 1 file and it **must** be a .pdf. www.combinepdf.com
- Read the exam protocol in the syllabus.

Reading Debrief:

- Discuss Section 10.3 (on second-order partials) w/ your group.
- Are there any questions we should address?

Activity 10.3.4

v\T	-30	-25	-20	-15	-10	-5	0	5	10	15	20
5	-46	-40	-34	-28	-22	-16	-11	-5	1	7	13
10	-53	-47	-41	-35	-28	-22	-16	-10	-4	3	9
15	-58	-51	-45	-39	-32	-26	-19	-13	-7	0	6
20	-61	-55	-48	-42	-35	-29	-22	-15	-9	-2	4
25	-64	-58	-51	-44	-37	-31	-24	-17	-11	-4	3
30	-67	-60	-53	-46	-39	-33	-26	-19	-12	-5	1
35	-69	-62	-55	-48	-41	-34	-27	-21	-14	-7	0
40	-71	-64	-57	-50	-43	-36	-29	-22	-15	-8	-1

$w(v,T)$

Symmetric Difference

$$f'(a) \approx \frac{f(a+h)-f(a-h)}{2h}$$

Estimate $w_T(20,-15)$. Use $f(T) = w(20,T)$

So
$$w_T(20,-15) \approx \frac{w(20,-15+h) - w(20,-15-h)}{2\cdot h} = \frac{-35 - (-48)}{10} = \frac{13}{10}$$

To estimate $w_{TT}(20,-10)$. Use $f(T) = w_T(20,T)$.

Then
$$w_{TT}(20,-10) \approx \frac{w_T(20,-10+5) - w_T(20,-10-5)}{10}.$$

Section 10.4 Tangent Planes and Differentials

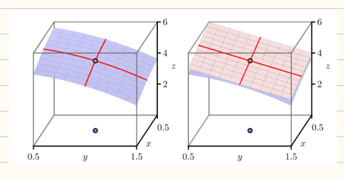
Definition A function $f(x,y)$ is continuously differentiable at (x_0,y_0) if f_x, f_y both exist and are continuous on an open disk containing (x_0,y_0)

This conditions ensures that the graph of the function is locally linear at (x_0,y_0) and therefore has a tangent plane.

Let's derive the eq of the tangent plane. Suppose

$$z = z_0 + a(x-x_0) + b(y-y_0)$$

is the tangent plane at the point $(x_0, y_0, f(x_0,y_0))$. By assumption $z_0 = f(x_0,y_0)$. Then



$$z_x(x_0,y_0) = a$$
$$z_y(x_0,y_0) = b$$

These quantities are the slopes of the tangent lines to the x and y traces of z. The x and y traces of f have precisely the same tangent lines. Therefore,

$$a = z_x = f_x(x_0,y_0)$$
$$b = z_y = f_y(x_0,y_0).$$

Tangent Plane If $f(x,y)$ is continuously diff. at (x_0,y_0), then the tangent plane to the graph of f at (x_0,y_0) exists and has scalar equation:

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

Activity 10.4.2

- Complete Activity 10.4.2 and discuss w/ your group.
- Class discussion.

a. $f(x,y) = 2+4x-3y$ at $(1,2)$.

$f_x(1,2) = 4$

$f_y(1,2) = -3$

⇒ Tangent plane

$$z = 0 + 4(x-1) - 3(y-2) = 4x - 3y + 2$$

Observation: the graph of f is the plane is the set of points

$$z = 2+4x-3y.$$

Conclusion: the tangent plane to a plane is the same plane!

b. $f(x,y) = x^2y$ at $(1,2)$.

$f_x(1,2) = 4$

$f_y(1,2) = 1$

$$\Rightarrow z = 2 + 4(x-1) + (y-2) = 4x + y - 4.$$

Section 10.4.2 Linearization

A tangent plane the graph of a functions provides a decent approximation of the function. When we think of the tangent plane in this way, we call it the linearization of the function and we write

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

Notice that

$$f(x,y) \approx L(x,y)$$

as long as (x,y) is "close to" (x_0,y_0).

Activity 10.4.3

- Complete Activity 10.4.3 and discuss w/ your group.
- Class discussion.

a. Find $L(x,y)$ for $g(x,y) = \frac{x}{x^2+y^2}$ at $(1,2)$.

$$g_x(x,y) = \frac{1(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad x(x^2+y^2)^{-1}$$

$$g_x(1,2) = \frac{3}{25}$$

$$g_y(1,2) = -x(x^2+y^2)^{-2}\cdot 2y = \frac{-2xy}{(x^2+y^2)^2} \qquad g_y(1,2) = \frac{-4}{25}$$

$$\Rightarrow L(x,y) = \frac{1}{5} + \frac{3}{25}(x-1) - \frac{4}{25}(y-2).$$

b. Find linearization $L(v,T)$ at $(20,-10)$.

$$L(v,T) = w(20,-10) + w_v(20,-10)(v-20) + w_T(20,-10)(T+10).$$

From table $w(20,-10) = -35$.

From before $w_v(20,-10) = -\frac{1}{2}$

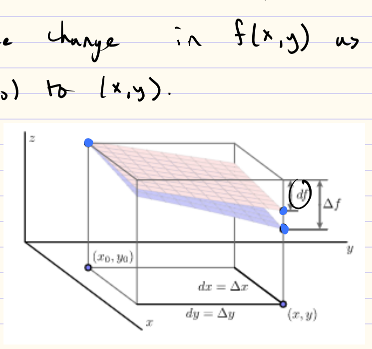
$$w_T(20,-10) = \frac{13}{10}.$$

$$L(v,T) = -35 - \frac{1}{2}(v-20) + \frac{13}{10}(T+10).$$

Then
$$w(10,-10) \approx L(10,-10) = -30$$
$$w(20,-12) \approx L(20,-12) = -37.6$$
$$w(18,-12) \approx L(18,-12) = -36.6.$$

Section 10.4.3 Differentials

The linearization $L(x,y)$ at (x_0,y_0) can be used to approximate the change in $f(x,y)$ as we move from the point (x_0,y_0) to (x,y).



Suppose we start at (x_0,y_0) and we move to (x,y). Then $(x,y) = (x_0+\Delta x, y_0+\Delta y)$. We want to measure is

$$\Delta f = f(x,y) - f(x_0,y_0).$$

From the picture, if $L(x,y)$ is the linearization of f at (x_0,y_0), then

$$\Delta L \approx \Delta f.$$

We write $df = \Delta L$ and call it a differential. Since $f(x_0,y_0) = L(x_0,y_0)$, we have

$$df = \Delta L = L(x,y) - L(x_0,y_0)$$
$$= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - f(x_0,y_0)$$
$$= f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y.$$

For consistency of notation we write $dx = \Delta x$ and $dy = \Delta y$ and call these differentials.

So the formula becomes

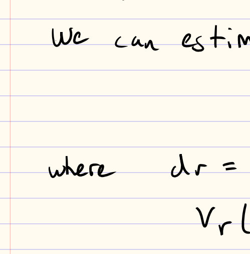
$$df = f_x\,dx + f_y\,dy$$

or

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

We think of df, dx, dy as small changes in x, y, and f. The equation just tells us how they are related.

Example Approximate the volume of tin in a cylindrical tin can w/ radius $r = 4cm$ and height $h = 12cm$ if the thickness of the tin is .04cm.



The volume of a cylinder w/ radius r and height h is given by

$$V(r,h) = \pi r^2 \cdot h$$

We want to estimate $\Delta V = V(4+.04, 12+.08) - V(4,12)$. We can estimate this using dV. We have

$$dV = V_r(4,12)\,dr + V_h(4,12)\,dh$$

where $dr = .04$ and $dh = .08$. We have

$$V_r(r,h) = 2\pi r h$$
$$V_h(r,h) = \pi r^2$$

So
$$dV = 2\pi\cdot 4\cdot 12\cdot .04 + \pi\cdot 16\cdot .08 \approx 16.08\ cm^3$$